1	0	0	0	0	0	0
1	1	0	0	0	0	0
1	2	1	0	0	0	0
1	3	3	1	0	0	0
1	4	6	4	1	0	0
1	5	10	10	5	1	0
1	6	15	20	15	6	1
1	7	21	35	35	21	7
1	8	28	56	70	56	28
1	9	36	84	126	126	84
1	10	45	120	210	252	210

$$\binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

apply this to…

$$\left(1+\frac{1}{n}\right)^n$$

$$1^n + \frac{n}{1}(1)^{n-1}\left(\frac{1}{n}\right)^1 + \frac{n\cdot(n-1)}{2\cdot 1}(1)^{n-2}\left(\frac{1}{n}\right)^2 +$$

$$\frac{n\cdot(n-1)\cdot(n-2)}{3\cdot 2\cdot 1}(1)^{n-3}\left(\frac{1}{n}\right)^3 +$$

$$\frac{n\cdot(n-1)\cdot(n-2)\cdot(n-3)}{4\cdot 3\cdot 2\cdot 1}(1)^{n-4}\left(\frac{1}{n}\right)^4 +\cdots$$

$$1+1+\frac{\left(1-\frac{1}{n}\right)}{2!}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3!}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)}{4!}+\cdots$$

As n approaches infinity,

$$1+1+\frac{\left(1-\frac{1}{n}\right)}{2!}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3!}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)}{4!}+\cdots$$

approximates

As n approaches infinity,

$$1+1+\frac{\left(1-\frac{1}{n}\right)}{2!}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3!}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)}{4!}+\cdots$$

approximates

$$1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots$$

$$\left(1+\frac{r}{n}\right)^n$$

$$1^n+\frac{n}{1}(1)^{n-1}\left(\frac{r}{n}\right)^1+\frac{n\cdot(n-1)}{2\cdot 1}(1)^{n-2}\left(\frac{r}{n}\right)^2+\frac{n\cdot(n-1)\cdot(n-2)}{3\cdot 2\cdot 1}(1)^{n-3}\left(\frac{r}{n}\right)^3+\frac{n\cdot(n-1)\cdot(n-2)\cdot(n-3)}{4\cdot 3\cdot 2\cdot 1}(1)^{n-4}\left(\frac{r}{n}\right)^4+\cdots$$

$$1+1r+\frac{\left(1-\frac{1}{n}\right)}{2!}r^2+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3!}r^3+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)}{4!}r^4+\cdots$$

approximates

$$1+r+\frac{r}{2!}+\frac{r^2}{3!}+\frac{r^3}{4!}+\cdots$$

How do $1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots$ and $1+r+\frac{r}{2!}+\frac{r^2}{3!}+\frac{r^3}{4!}+\cdots$ compare?

$1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots$ comes from $\left(1+\frac{1}{n}\right)^n$

$1+r+\frac{r}{2!}+\frac{r^2}{3!}+\frac{r^3}{4!}+\cdots$ comes from $\left(1+\frac{r}{n}\right)^n$

But remember that $\left(1+\frac{1}{n}\right)^r =$

$$1^n + \frac{r}{1}(1)^{n-1}\left(\frac{1}{n}\right)^1 + \frac{r\cdot(r-1)}{2\cdot 1}(1)^{n-2}\left(\frac{1}{n}\right)^2 + \frac{r\cdot(r-1)\cdot(r-2)}{3\cdot 2\cdot 1}(1)^{n-3}\left(\frac{1}{n}\right)^3 + \frac{r\cdot(r-1)\cdot(r-2)\cdot(r-3)}{4\cdot 3\cdot 2\cdot 1}(1)^{n-4}\left(\frac{1}{n}\right)^4 + \cdots$$

which is equivalent to $1+\frac{r}{n}+\frac{\frac{r}{n}\left(\frac{r}{n}-\frac{1}{n}\right)}{2!}+\frac{\frac{r}{n}\left(\frac{r}{n}-\frac{1}{n}\right)\left(\frac{r}{n}-\frac{2}{n}\right)}{3!}+\frac{\frac{r}{n}\left(\frac{r}{n}-\frac{1}{n}\right)\left(\frac{r}{n}-\frac{2}{n}\right)\left(\frac{r}{n}-\frac{3}{n}\right)}{4!}+\cdots$

which is approximately $1+\frac{r}{n}$ for large n

So, $\left(\left(1+\frac{1}{n}\right)^r\right)^n$, or $\left(1+\frac{1}{n}\right)^{nr}$ is an approximation of $\left(1+\frac{r}{n}\right)^n$

What if we extended Pascal's triangle upwards?

1	0	0	0	0	0	0
1	1	0	0	0	0	0
1	2	1	0	0	0	0
1	3	3	1	0	0	0
1	4	6	4	1	0	0
1	5	10	10	5	1	0
1	6	15	20	15	6	1

What if we extended Pascal's triangle upwards?

1	0	0	0	0	0	0
1	1	0	0	0	0	0
1	2	1	0	0	0	0
1	3	3	1	0	0	0
1	4	6	4	1	0	0
1	5	10	10	5	1	0
1	6	15	20	15	6	1

What if we extended Pascal's triangle upwards?

1	-1	1	-1	1	-1	1
1	0	0	0	0	0	0
1	1	0	0	0	0	0
1	2	1	0	0	0	0
1	3	3	1	0	0	0
1	4	6	4	1	0	0
1	5	10	10	5	1	0
1	6	15	20	15	6	1

What if we extended Pascal's triangle upwards?

1	-2	3	-4	5	-6	7
1	-1	1	-1	1	-1	1
1	0	0	0	0	0	0
1	1	0	0	0	0	0
1	2	1	0	0	0	0
1	3	3	1	0	0	0
1	4	6	4	1	0	0
1	5	10	10	5	1	0
1	6	15	20	15	6	1

1	-3	6	-10	15	-21	28
1	-2	3	-4	5	-6	7
1	-1	1	-1	1	-1	1
1	0	0	0	0	0	0
1	1	0	0	0	0	0
1	2	1	0	0	0	0
1	3	3	1	0	0	0
1	4	6	4	1	0	0
1	5	10	10	5	1	0
1	6	15	20	15	6	1

1	-4	10	-20	35	-56	84
1	-3	6	-10	15	-21	28
1	-2	3	-4	5	-6	7
1	-1	1	-1	1	-1	1
1	0	0	0	0	0	0
1	1	0	0	0	0	0
1	2	1	0	0	0	0
1	3	3	1	0	0	0
1	4	6	4	1	0	0
1	5	10	10	5	1	0
1	6	15	20	15	6	1

1	-4	10	-20	35	-56	84
1	-3	6	-10	15	-21	28
1	-2	3	-4	5	-6	7
1	-1	1	-1	1	-1	1
1	0	0	0	0	0	0
1	1	0	0	0	0	0
1	2	1	0	0	0	0
1	3	3	1	0	0	0

$$1 - x + x^2 - x^3$$

$$x + 1 \overline{)\,1}$$

$$\underline{1 + x}$$

$$-x$$

$$\underline{-x - x^2}$$

$$x^2$$

$$\underline{x^2 + x^3}$$

$$-x^3$$

$$\underline{-x^3 - x^4}$$

$$x^4$$

$1-x+x^2-x^3+x^4-x^5\cdots$ converges to $\dfrac{1}{x+1}$ if $0<x<1$.

Or from the sum of a geometric series: $\dfrac{1}{1-r}$

if $r = -x$, then you have $1-x+x^2-x^3+x^4-x^5\cdots$

DELVING deeper

Debra K. Borkovitz

What's So Special about 3?