Deriving the Poisson Distribution from the Binomial Distribution

Andrew Chamberlain, Ph.D. Follow
Dec 10, 2016 · 5 min read

At first glance, the binomial distribution and the Poisson distribution seem unrelated. But a closer look reveals a pretty interesting relationship.

https://medium.com/@andrew.chamberlain/deriving-the-poisson-distribution-from-the-binomial-distribution-840cc1668239

Define a number

$$\lambda = np$$

Let this be the rate of successes per day. It's equal to np. That's the number of trials n—however many there are—times the chance of success p for each of those trials.

Think of it like this: if the chance of success is p and we run n trials per day, we'll observe np successes per day on average. That's our observed success rate lambda.

Recall that the binomial distribution looks like this:

$$B(p, n) = P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

As mentioned above, let's define lambda as follows:

$$\lambda = np$$

Solving for p, we get:

$$\Rightarrow p = \frac{\lambda}{n}$$

What we're going to do here is substitute this expression for p into the binomial distribution above, and take the limit as n goes to infinity, and try to come up with something useful. That is,

$$\lim_{n\to\infty} P(X = k) = \lim_{n\to\infty} \frac{n!}{k!(n-k)!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

Pulling out the constants

$$\lambda^k$$

and

$$\frac{1}{k!}$$

and splitting the term on the right that's to the power of (n-k) into a term to the power of n and one to the power of -k, we get

$$\left(\frac{\lambda^k}{k!}\right) \lim_{n\to\infty} \frac{n!}{(n-k)!} \left(\frac{1}{n^k}\right) \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-k}$$

Now let's take the limit of this right-hand side one term at a time. We'll do this in three steps. The first step is to find the limit of

$$\lim_{n\to\infty} \frac{n!}{(n-k)!}\left(\frac{1}{n^k}\right)$$

In the numerator, we can expand n! into n terms of (n)(n-1)(n-2)…(1). And in the denominator, we can expand (n-k) into n-k terms of (n-k)(n-k-1)(n-k-2)…(1). That is,

$$\lim_{n\to\infty} \frac{n!}{(n-k)!}\left(\frac{1}{n^k}\right)$$

In the numerator, we can expand n! into n terms of (n)(n-1)(n-2)…(1). And in the denominator, we can expand (n-k) into n-k terms of (n-k)(n-k-1)(n-k-2)…(1). That is,

$$\lim_{n\to\infty} \frac{n(n-1)(n-2)...(n-k)(n-k-1)...(1)}{(n-k)(n-k-1)...(1)}\left(\frac{1}{n^k}\right)$$

Written this way, it's clear that many of terms on the top and bottom cancel out. The (n-k)(n-k-1)…(1) terms cancel from both the numerator and denominator, leaving the following:

$$\lim_{n\to\infty} \frac{n(n-1)(n-2)...(n-k+1)}{n^k}$$

Since we canceled out n-k terms, the numerator here is left with k terms, from n to n-k+1. So this has k terms in the numerator, and k terms in the denominator since n is to the power of k.

Expanding out the numerator and denominator we can rewrite this as:

$$\lim_{n\to\infty}\left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\ldots\left(\frac{n-k+1}{n}\right)$$

This has k terms. Clearly, every one of these k terms approaches 1 as n approaches infinity. So we know this portion of the problem just simplifies to one. So we're done with the first step.

The second step is to find the limit of the term in the middle of our equation, which is

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n$$

Hat Check Problem

In a restaurant n hats are checked and they are hopelessly scrambled; what is the probability that no one gets his own hat back?

Find the probability that a random permutation contains at least one fixed point.

https://math.dartmouth.edu/archive/m60s06/public_html/Lecture8.pdf

- If A_i is the event that the ith element a_i remains fixed under this map, then

$$P(A_i) = \frac{1}{n}.$$

https://math.dartmouth.edu/archive/m60s06/public_html/Lecture8.pdf

- If we fix a particular pair (a_i, a_j), then

$$P(A_i \bigcap A_j) = \frac{1}{n(n-1)}.$$

- The number of terms of the form $P(A_i \bigcap A_j)$ is $\binom{n}{2}$.

https://math.dartmouth.edu/archive/m60s06/public_html/Lecture8.pdf

- For any three events A_1, A_2, A_3

$$P(A_i \cap A_j \cap A_k) = \frac{(n-3)!}{n!} = \frac{1}{n(n-1)(n-2)} \, ,$$

and the number of such terms is

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{3!} \, .$$

https://math.dartmouth.edu/archive/m60s06/public_html/Lecture8.pdf

- Hence

$$P(\text{at least one fixed point}) = 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots (-1)^{n-1}\frac{1}{n!}$$

https://math.dartmouth.edu/archive/m60s06/public_html/Lecture8.pdf